THE USE OF NEGATIVE PROBABILITIES IN ECONOMICS

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Abstract

The concept of „negative probability" has become one of interest in research. Initially developed in connection with physics (negative probabilities are important in the analysis of quantum phenomena), nowadays it has applications in many other scientific disciplines. Its main advantage is that it extends the classical probability theory, which involves values between 0 and 1, allowing negative or supraunitary values. From this point of view, negative probabilities are a useful tool that facilitates the calculation and makes analysis more flexible through models. This paper aims to present the theoretical framework, the principles in relation to which the use of negative probabilities in economics, in general, can be understood. The existing literature highlights that the negative probabilities are appropriate in the analysis of stochastic processes and, from this perspective, has developed applications especially in the field of finance. In this sense, negative probabilities are used in models such as the binomial CRR model (The Cox-Ross-Rubinstein Market Model), that evaluate the price of derivatives on the capital market, those that include hidden variable (non-observable) etc. In our analysis we describe these models and, based on them, we try to identify the features of the negative probabilities that can be used in the economic field. The research is a theoretical one, leading the way for new approaches in the direction of using negative probabilities in macroeconomic modelling.

Keywords: *negative probabilities, economics, CRR model, negative interest*

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1. Introduction

The concept of *negative probability* is relatively a controversial one. Although it has attracted the attention of researchers over time and developed applications in several fields of activity (physics, economics, engineering), for a part of the scientific world, which approaches probabilities in the traditional way, closely related to the perception that the chance of an event can only be found in the range [0,1], it's seen as an anomaly that implies the existence of errors in the models in which it occurs. In this paper we analyze the theoretical framework, the principles in relation to which the use of negative probabilities in economics, in general, can be understood. As shown in the few specialized papers that have dealt with the subject, it is especially suitable in the analysis of stochastic processes and, from this perspective, has developed applications mostly in the field of finance. The aim of the paper is not to present new discoveries related to this issue but to highlight, in a unitary perspective, a series of aspects that can help to understand how far it reached in treating it. Thus, beyond this introduction, in Chapter 2 we analyzed the concept of probability in general, highlighting the features of the Kolmogorov approach, in Chapter 3 the negative probability, focusing on the main theories on it, and in Chapter 4 the way they can be used in economics. Chapter 5 revealed the main conclusions of the paper.

2. Probability. General issues

The concept of probability comes from the Latin (lat. *probare* - to prove, respectively *ilis* - property) and means the property to be verified. In a general, objective, frequency-based approach, it reveals the chance of an event to occur and is calculated as the *total number of outcomes in the total possible ones*. Beyond this, there are other interpretations of probabilities, such as the subjective, epistemic, logical, intuitive etc.

The theory that has prevailed in the field of probabilities is, however, that of Kolmogorov. Developed in the 1930s, it is based on the *theory of measure* and takes into account the notions of *probability space* of a random event and *random variable*. Related to these, the Kolmogorov's axioms have been developed, which we present below.

2.1. Kolmogorov's axioms

Let's consider a measure space, (Ω, F, P), and *P(E)*, the probability of the event *E*, where *P(Ω)* = 1. Thus, (Ω, F, P) is a probability space *A*, with Ω being a set, F, a subset of events, and P, a probability function (Scalas, 2008).

- **Axiom 1.** $\boldsymbol{P(A) \geq 0}$**;**
- **Axiom 2.** $\boldsymbol{P(\Omega)} = \mathbf{1}$ (the probability of a certain event is equal to 1);
- **Axiom 3.** $\boldsymbol{P(A \cup B)} = \boldsymbol{P(A)} + \boldsymbol{P(B)}$, if $\boldsymbol{A \cap B} = \emptyset$;

2.2. Consequences of the Kolmogorov's axioms

In relation with the above, the following consequences of probabilities arise:

- **Probability of the empty set.**
  $P(\emptyset) = 0$;
- **Probability of complement**. For every A in the subset F, we have:
  $P(A^c) = 1 - P(A)$;
- **Probability of union.** For each couple of sets A and B in subset F, we have:
  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

The main conclusion that emerges from Kolmogorov's axioms, in the context of our analysis, is that the probability of an event can be only positive and subunit, belonging to the interval [0,1].

3. Negative probabilities

The interest in *negative probabilities* arose and developed in connection with the field of physics and mainly quantum mechanics. Subsequently, were developed applications in economics, engineering and others. All these areas allow working with quasiprobabilities, implying the existence of hidden variables etc.

The first to notice the existence of negative probabilities was Wigner (1932), when he discovered a probability function similar to conventional ones that could result in negative values (*Wigner quasiprobability*). This quasiprobability was mainly used to analyze how quantum corrections work in mechanics. The term negative probability itself was later introduced by another physicist, P. Dirac (1942). In his work entitled *"The Physical Interpretation of Quantum Mechanics"* he showed that (similar to negative energies) these should not be considered absurd but understood in the sense that the negative money works. This, as well as the existence of supraunitary probabilities, were later proved by Feynman (1987) using a series of concrete examples with conditional probabilities. It showed that as long as the final result is positive, negative probabilities can be useful tools in calculations, working like negative numbers.

Beyound the above, there have been other contributions that have shown that negative probabilities can help solve multiple, complex problems, such as that of half of the coin (Székely, 2005) and others. A feature of all this, they related to the negative probabilities as a useful instrument in calculations and less tried to explain their nature.

From this last point of view, among the few contributions that developed theoretical frameworks were those of Khrennikov (1999), Burgin (2010), Blass (2020) etc.

3.1. Khrennikov and the p-adic theory of probabilities

The mathematician A. Khrennikov (1999) was the first to develop a theory that explain the causes and provide principles underlying negative probabilities. Among the reasons why it don't appear in Kolmogorov's approach, he points out, are the emphasis on *ensemble distribution* and the relationship between probability and *frequency*. These make the probabilities work in the traditional metric and allow to obtain values exclusively between [0,1].

In this context, shows Khrennikov (1999), we can acquire negative probabilities extending the analysis beyond the conventional approaches. Such a perspective may be, in his opinion, the work of negative probabilities in connection with the so-called *p-adic numbers*. These involve the use of complex numbers, outside the usual one, based on rational numbers, in a specific framework, by integrating the notion of *closeness*.

An important aspect, this is somewhat a limited proposal, as it can only be applied in the context of the *p-adic numbers*, but it offers an explanation regarding the negative probabilities.

3.2. Extended probability theory

A theory of negative probabilities from the perspective of using real numbers is developed by Burgin (2010). It tries to highlight their features and starts from the hypothesis of the existence of an extended field of probability, in which they can take both positive and negative values. His theory, which we describe below, is a frequency-based reinterpretation of probabilities.

In this context, we have:

- a set Ω – having as components the subset Ω^+ and the subset Ω^-;
- a set F including elements of Ω;
- a probability function P, with values in R (real numbers);

In the sense of the above, let's take into account the following definitions:

- elements of the set F - *random events*;
- elements of the set F^+ (with $X \in F$ and $X \subseteq \Omega^+$) - *positive random events*;
- elements of the set Ω^+ belonging to F^+ - *elementary positive random events*.

Now, if we consider an element $w \in \Omega^+$, then we can have $-w$, an *antievent* of *w*. In this context, can be called *random elementary antievents (or negative random elementary elements)*, elements in Ω^- belonging to F^-.

Thus, in a more general perspective, for any set $X \subseteq \Omega^+$ we have:

$X+ = X \cap \Omega^+$;

$X^- = X \cap \Omega^-$;

$-X = \{-w; w \in X\}$, respectively

$F^- = \{-A; A \in F^+\}$,

with $-A$, the *antievent* of *A*, and the elements in F^-, *negative random events (*or *random antievents)*.

3.2.1. Extended frequency probabilities

Now, given the traditional definition of probabilities, based on relative frequencies, let's take an event $\{u_i\}$, with $u_i \in \Omega$, and we have:

$\boldsymbol{v_N (u_i) = (n_i)/N^+ - (m_i)/N^-}$, where

N^+ – the incidence of events **with a similar sign to u_i** in N trials;

N^- – the incidence of events **with the opposite sign to u_i** in N trials;

n_i – the incidence of event *ui* in N trials;

m_i – the incidence of event *-ui* in N trials;

In the context of the above, the ***extended frequency probability*** becomes:

$\boldsymbol{p(u_i) = \lim_{N\to\infty} v_N(u_i)}$,

where $\boldsymbol{u_i} = \boldsymbol{w_i}$, or $\boldsymbol{u_i} = \boldsymbol{-w_i}$, with the feature:

$\boldsymbol{p(wi) = - p(-wi)}$ (the probability of the event is equal to the *minus* probability of the *antievent*).

In a more general case, when we consider a random event A = {w_{i1}, w_{12},…, w_{ik}, -w_{j1}, -w_{j2}…-w_{jt}}, the relationship that defines the probability of the extended frequency becomes:

$$\boldsymbol{p(A) = \lim_{N\to\infty} \nu_N(A).}$$

3.2.2. Axioms of extended probabilities

Considering the above, a **probability function,** as a function noted with P of an event belonging to F, with values in R (real numbers) is subscribed to the axioms (Burgin, 2010):

- **First Axiom:** There is α: Ω → Ω i.e. α^2 is a identity mapping on Ω having the features: α(*w*) = -*w* for any *w* in Ω, α(Ω$^+$) ⊆ Ω$^-$, and if *w* ∊ Ω$^+$, then α(*w*) ≠ Ω$^+$;
- **Second Axiom:** There is a set algebra F$^+$ ≡ {X ∊ F; X ⊆ Ω$^+$}, where Ω$^+$ is an element;
- **Third Axiom:** P(Ω$^+$) = 1;
- **Fourth Axiom:** F ≡ {X; X$^+$ ⊆ F$^+$ and X$^-$ ⊆ F- and X$^+$ ⋂ -X$^-$ ≡ ∅ and X$^-$ ⋂ -X$^+$ ≡ ∅};
- **Fifth Axiom:** $\boldsymbol{P}(\boldsymbol{A}\cup\boldsymbol{B}) = \boldsymbol{P}(\boldsymbol{A}) + \boldsymbol{P}(\boldsymbol{B})$, with A, B ∊ F, and $\boldsymbol{A}\cap\boldsymbol{B}=\emptyset$;
- **Sixth Axiom:** There is $\{v_i,\ w,\ -w;\ v_i$, w ∊ Ω and i ∊ I} = $\{v_i;\ v_i$, w ∊ Ω and i ∊ I};
- **Seventh Axiom:** If A = B, then P(A) = P(B), where A and B ∊ F; P({w,-w}) = P(∅) = 0;
- **Eighth Axiom:** P(A) ≥ 0**,** where A ∊ F$^+$;
- **Ninth Axiom:** P(A) = P(A$^+$) + P(A$^-$), where A ∊ F.

4. Negative probabilities and their use in economics

In economics, negative probabilities have developed applications mainly in terms of risk modeling, in quantitative finance, and less in other subdomains. The explanation is given by the fact that here the variables imply quasiprobabilities (neutral risk is an example) and less real probabilities. The main papers that have dealt with this topic so far are those of Haug (2004) and Burgin (2011). The first focuses on the CRR binomial model, used to calculate the price of derivatives and shows that negative probabilities occur naturally within it. The latter reveals how negative probabilities can be used in modeling the interest rate, when it records values with the *minus* sign. Both show how the appeal to negative probabilities can be a solution in solving problems in models characterized by dysfunctions, the existence of hidden variables, the occurrence of unexpected phenomena etc. These two case studies will be presented in the following.

4.1. The CRR binomial model

The CRR binomial model (Cox, Ross and Rubinstein, 1979) is used to determine the prices of derivatives in different time periods. The model, a network type, implies the existence of several nodes, at each of which the prices fluctuate up, depending on a factor *u*, or down, depending on a fixed factor *d*. If we describe the whole stochastic process as a continuous Brownian motion, we obtain:

$$dS = \mu S dt + \sigma S dZ \qquad \text{Equation (1)}$$

where S – the asset price; μ – drift coefficient; and σ – diffusion coefficient (volatility).

The price, in each node, defined depending on the initial value $S_0 > 0$, through a Bernoulli process, is: $S_0 u^i d^{j-i}$, where i = 0, 1, 2,… , j,

and the size of the up and down motion, in each step, is

$$u = e^{\sigma\sqrt{\Delta t}},\ d = \frac{1}{u} = e^{-\sigma\sqrt{\Delta t}} \qquad \text{Equation (2)}$$

with $u \geq 1$, $0 < d \leq 1$, respectively $\Delta t = T/n$ - the size of steps, n - steps in time.

Now, P probability, considered a risk-neutral, the set of probabilities in the different nodes, can be integrated into the CRR model, for each of them, in the context of the arbitrage price, with the formula:

$$Se^{ri\Delta t} = p_i uS + \left(1 - p_i\right) dS \qquad \text{Equation (3)}$$

where p_i, a neutral-risk probability (this is a quasiprobability), respectively r_i – the asset return.

Thus, if we solve the equation, we obtain:

$$p_i = \frac{e^{ri\Delta t} - d}{u - d}$$ Equation (4)

In this context, the going down probability of the derivative price is equal to ***1 – pi***, which makes that when ***pi* > 1**, this to become negative. This may be possible, as Haug (2004) shows, when low volatility *σ* and high *r* are recorded.

4.2. Interest rate modeling

Regarding the interest rate, one way to model it is also based on the Brownian motion, showing its continuous evolution over time. This model is shaped as follows:

$$\frac{dr}{r} = \mu dt + \sigma\varepsilon\sqrt{dt}$$ Equation (5)

where μ - drift coefficient, dt – short time period, σ - diffusion coefficient (volatility), *ε* – the stochastic element.

In this context, the log-normal distribution of *r*, as a probability density function (pdf), will be as follows:

$$\frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{ln(x) - \mu}{\sigma}\right)^2}$$ Equation (6)

Similar to the previous example, the main problem in the model appears when the interest rate takes negative values (which is possible, the concept of negative interest rate exists), because the logarithm of a non-positive number can't be calculated. In this case the equation can be solved by using negative probabilities (see Burgin, 2011).

5. Conclusions

Closely related to those discussed above, we present the conclusions of this paper, as follows:

- Negative probabilities are not "anomalies" in models, they *can occur naturally*, similar to conventional probabilities;
- In principle, negative probabilities can occur in subsets of events; however, the probability of the set consisting of these subsets is positive;

- When we have negative probabilities, the problem is not of the models in which they are obtained, but of the *space (or metrics)* chosen to be used (for example, p-adic analysis and extended probabilities makes it possible to work with such negative values);
- Negative probabilities are useful tools because they can add *flexibility* to models;
- In economics, the use of negative probabilities is possible because many of them are quasiprobabilities (neutral risk probabilities etc.);
- Negative probabilities can be used in economics when we have: negative interest rates, hidden variables (materializing or not but which can influence observable variables), as the expected return, volatility etc.

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