IDENTIFICATION OF THE PRODUCTION FUNCTION BY THE FORM OF THE MARGINAL CHARACTERISTICS, THE MARGINAL SUBSTITUTION RATE, THE ELASTICITIES AND THE COST FUNCTION.

Alexandru BRAILA[1], **Zinovia TOACA**[2]

Academy of Economic Studies of Moldova

Abstract

This communication proposes some ways of building production functions (PF) based on its assumptions, characteristics and main properties. The specificity of the activities of each production system leads to the idea that there is a wide variety of PF, virtually every system having its own production function. This is not so much as regards the parameters of the various types of PF as the shape of these functions. Thus, depending on the particularities of those activities, the form of PF may be specified, a form which makes it possible to assess that it belongs to a certain class of functions mentioned, including certain amendments. The conceptual framework for identifying the form of the production function, which depends on the relative indicators of PF, is presented. It was analyzed induced PF form of marginal yields, elasticities, marginal substitution rate and substitution elasticity. Approval of proposed approaches can be carried out on the basis of statistical data using econometric methods.

Keywords*: Production function Cobb Douglas, marginal proprieties, elasticity, cost function.*

JEL Classification: *C01, C02, C13, C52, C53*

1. Introduction

The use of production functions (FP) in the process of economic analysis and forecasting continues to be an imperative of the time. Their use is important both at the microeconomic and macroeconomic level. Modifications

[1] *Associate Professor, Ph. D. ASEM Chisinau, Moldova, e-mail: braila.alşexandru.mihai@ase.md*
[2] *Associate Professor, Ph. D. ASEM Chisinau, Moldova, e-mail: toaca@ase.md*

have been made to the classical form of FP Cobb-Douglas, which allow determining a more adequate relationship between the inputs and outputs of the technological production process. The correctness of the estimated function is largely determined by its specification, deduced theoretically, that must be tested econometrically.

2. The scientific approach of the topic in the literature

The cyber approach involves examination of a production complex as *an open system* (inputs being human and material *resources expenditure*, and outputs – *production*). Starting from this approach (Ашманов С.А, 1984, p. 210-241), PF can be defined as an econometric model expressing *a stable quantitative ratio between inputs and outputs*. In a general PF form can be represented by equality $F(X, YA) = 0$, in which $Y = \left(y_{1,} y_2, \dots, y_m\right)$ is the vector of the production - outputs, $X = (x_1, x_2, \dots, x_n)$ is the vector of consumption of production factors – inputs into the production system, and $\boldsymbol{A}$ – parameter matrix. However, the following expression is often used for PF: $F(X)$=0, $X = (x_1, x_2, \dots, x_n)$, in wich the variables $x_i \leq 0$ designate inputs, $x_j \geq 0$ – production (outputs).

Usually, in economic research, PF has a restricted meaning (Intriligator M., 1989, p. 237-249), namely in the form of a single equation. All production components being brought together (value or natural expression) in a single scalar size (Y), and the number of heterogeneous production resources is reduced to a minimum (as a rule two factors) that allows the estimation of PF parameters on the basis of existing statistical information:

$$Y = F(x_1, x_2, \dots, x_n), < 10; or\ Y = F(K, L), \qquad (1)$$

where K – material resource called capital, L- human source (average number of persons employed during the period considered, days-work; hours-work, etc.).

As the quantitative connection between the costs and the results of the production has a statistical character, the FP represents an *econometric model* (Berndt Ernst R, 2005, p. 78-109; Rezagholi M., 2006, p.12-26). If the expenses x_i are used as exogenous variables, then the model is called, as a rule, a production function, and if the size of the output Y (is an exogenous variable) is fixed, then the model is called a *cost function*.

PF can be built (Gordon David, 2011) for a firm, a branch of the national economy and the national economy as a whole. The degree of aggregation of data may also be different, from a detailed nomenclature to indicators with a high degree of generalization.

Adequate reflection, using the production function model, of the real relationship between input resources (inputs) and output production (outputs) is broken down into two problems related to each other:

1. Identification (specification) PF, i.e. identification of the essential factors of the PF model and definition of the function form.

2. The parameterization of FP, that is, the calculation of the numerical values of the parameters with the help of systematized statistical data based on the econometric analysis.

The PF can be constructed under static (synchronous) appearance, based on a set of indicators at a single moment (cross-sectional data) and/or dynamic (Time-series data).

The first successful attempt to build a PF, was made by mathematician Cobb and economist Douglas (1928, USA). The PF form proposed by these scientists, is applied traditionally to this day due to its rational character and simplicity. The calculation of the parameters was made for the US manufacturing industry, based on statistical data for the period 1899-1922. Subsequently, PF Cobb-Douglas (CD), which verifies all logical, economic and mathematical requirements, was generalized under different aspects. First, it was found that this need to more accurately reflect the effect of the dimensions of the production and the influence of the technical progress (Solow R, 1957), for which purpose the respective modifications were made. The sum of the exponents of the powers (Dong Hag, 2014 p. 312-349) of the factors K and L does not necessarily have to be equal to one, a multiplier of *technical progress* is introduced.

PF must check certain logical, economic and mathematical requirements:

– all sizes that are included in the PF must be measurable;

– production is impossible without resource expenditure;

– all resources contained in the PF are necessary (this condition is not always respected);

– among the PF arguments must be included the essential factors for the realization of the respective production process (obviously this condition is not univocal);

– it is assumed that resources are to some extent interchangeable (substitutable); in the limit cases they can be complementary, that is to say they can enter strictly determined proportions;

– PF must have an appropriate statistical justification;

– variables are continuous (that do not always correspond to reality);

These main requirements (their list can be extended (Ашманов, 1984 p. 210-241)) essentially "reduce" the classes of functions that can be PF - linear production functions - less common case, Cobb-Douglas, type CES, VES, with fixed proportion of resources - type Leontief and certain modifications thereof.

3. Conceptual framework for identifying (specifying) the form of the production function

The specificity of the activities of each production system leads to the idea that there is a wide variety of PF, virtually every system having its own production function. This is not so much in terms of parameters of different types of PF – the values of parameters, which are estimated by econometric methods (e.g. The Method of the Ordinary Lest Squares (OLS)) as to the form of these functions, which can be specified by the different methods (modalities). Traditional (in practice), the classic Cobb-Douglas function is most often used ($Y = F(K, L) = AK^{\alpha}L^{1-\alpha}$) or neoclassical („without technical progress”:$Y = AK^{\alpha}L^{\beta}, \alpha + \beta \neq 1$; „with technical progress”: $Y = AK^{\alpha}L^{\beta}e^{\lambda t}, etc.$), although its choice (Рузанова А. И., 2013) does not always have the basis on the specifics of the production activities of the manufacturing system. Therefore, in this publication, we intend to find according to the particularities of these activities the FP form by which they can be specified, a form that allows the assessment that it belongs to a certain class of functions mentioned, including with certain modifications.

The proposed method (method) for identifying PF for a certain production system (firm or aggregate: at branch or national economy) is based on phenomenological research: statistical determination of the most stable correlations between relative indicators (PF characteristics: average characteristics, marginal, percentages (elasticities)) and on this basis, the deduction of PF by the algorithm:

- **Step 1:** A stable correlation between relative indicators shall be determined. This correlation is expressed by a differential equation in which, as a rule, the dependent variable (endogenous) is the productivity of work.
- **Step 2:** This equation is integrated and the PF is obtained.

The method of identifying the PF approached allows to diversify the classes of functions which further verify the assumptions relating to PF and thus extend the possibilities of more successful choice of PF form appropriate to the manufacturing process investigated.

4. The main characteristics (relative indicators) of PF. It will be examined the PF $Y = F(K, L)$, homogeneous with a degree of homogeneity equal to *m*. In this case, it is natural to switch to new variables: $k = \frac{K}{L}$ - endowment of work with material resources (capital) and average labor productivity -$y = \frac{Y}{L}$.

From the homogeneity relationship: $F(\lambda K, \lambda L) = \lambda^m F(K, L)$, taking $\lambda = \frac{1}{L}$, it is obtained:

$$F\left(\frac{K}{L}, 1\right) = F(k, 1) \equiv f(k) = L^{-m} F(K, L), \tag{2}$$

and so $F(K, L) = L^m f(k)$. (3)

Hence result the main features:

Average features of PF:

a) the average productivity of labor:

$$APL(k) = y = L^{m-1} f(k); \tag{4}$$

b) the average productivity of capital product (average return on capital):

$$APK(k) = L^{m-1} \frac{f(k)}{k}. \tag{5}$$

Marginal features of PF:

a) marginal productivity of labor:

$$MPL(k) = L^{m-1}\big(mf(k) - kf'(k)\big); \tag{6}$$

b) marginal product of capital (marginal return of capital):

$$MPK(k) = L^{m-1} f'(k). \tag{7}$$

Marginal rate of substitution of labor with capital (MRS):

$$MRS(k) = -\frac{dK}{dL} = \frac{MPL}{MPK} = m\frac{f(k)}{f'(k)} - k \tag{8}$$

Percentage features (elasticities of inputs):

a) Elasticity of the labor factor:

$$E_L(Y) \equiv EL(k) = \frac{MPL}{APL} = m - k\frac{f'(k)}{f(k)} \tag{9}$$

b) Elasticity of the capital factor:

$$E_K(Y) \equiv EK(k) = \frac{MPK}{APK} = k\frac{f'(k)}{f(k)}. \tag{10}$$

c) Scale elasticity (total elasticity (ET) equal to homogeneity (m)):

$$ET = EK(k) + EL(k) = m. \quad (11)$$

d) Technical replacement elasticity (of work with capital)

$$\sigma = \frac{\frac{dk}{d(MRS)} * MRS}{k} = \frac{MRS}{k * MRS'(k)}\,. \quad (12)$$

The following will be presented some variants regarding the statistical specification of the possible stable correlation of the relative indicators (mentioned characteristics) and variable (*k*). Dependencies can be diverse: linear or nonlinear (parabolic, hyperbolic, logarithmic, exponential, etc.). Only linear dependencies will be analyzed.

Production functions induced by the form of marginal yields. It will start from the optimization problem: the choice of the resource combination (*K*, *L*) which ensures the minimum value of the production cost ($r * K - w * L \to min$), which leads us to equivalent equilibrium conditions: $\frac{MPK(k)}{r} = \frac{MPL(k)}{w} = \lambda$, (here *r*-unit cost of capital (rent), *w*-unit cost of labor (nominal salary) and λ is Lagrange's multiplier). It is obtained that the marginal indicators are proportional to the sizes *r* and *w*, which are in agreement with the average returns of the factors *K* and *L*. So marginal yields are dependent on average yields:

$$MPK = r(APK), like\ MPL = w(APL). \quad (13)$$

In case of linear dependence $r(APK) = a + bAPK$ and $w(APL) = a + bAPL$, differential equations are obtained:

$$L^{m-1}f'(k) = a + bL^{m-1}\frac{f(k)}{k} \quad (14)$$

a) $L^{m-1}\left(mf(k) - kf'(k)\right) = a + bL^{m-1}f(k).$ (15)

Solving the first equation identifies the PF of the form:

$$F(K, L) = \frac{a}{1-b}K + AK^{b}L^{m-b}. \quad (16)$$

IF $b = 0, m = 1$ - the obtained PF represents a linear production function, and if $a = 0$ - PF type Cobb-Douglas, for $\mathrm{m} = 1$ - PF is called classic.

Solving the differential equation (17), gets the next PF:

$$F(K, L) = \frac{a}{b-m}L + AK^{m-b}L^{b}, \quad (17)$$

which also represents a linear combination of variable *L* and PF type Cobb-Douglas.

Production functions induced by elasticity forme. For linear forme:

$$EK(k) = a + bk, \quad (18)$$

the differential equation is obtained:

$$k\frac{f'(k)}{f(k)} = a + bk, \quad (19)$$

the solution which leads us to the next form of PF:

$$F(K, L) = AK^a L^{m-a} e^{bk}, \quad (20)$$

which for $b = 0$ is PF type classical Cobb-Douglas (on the condition that $m = 1$) and therefore with constant *EK* and *EL* elasticities (fixed) and with $ET = m \neq 1$. The *m* parameter means the scale effect.

Production functions induced by the form of the marginal substitution rat*e*. If

$$MRS(k) = a + bk, \quad (21)$$

the differential equation is obtained:

$$m\frac{f(k)}{f'(k)} - k = a + bk, \quad (22)$$

that the solution leads us to the next form of PF:

$$F(K, L) = AL^{\frac{mb}{1+b}}\left((1 + b)K + aL\right)^{\frac{m}{1+b}}. \quad (23)$$

If $b = 0$, the situation corresponding to the hypothesis that MRS is constant, reflecting the economic content that at the optimum (maximum profit criterion):

$$MRS = \frac{MPL}{MPC} = \frac{w}{r}, \quad (24)$$

is constant and therefore the perfect indexation of the costs of inflation-rate factors occurs. In reality, wages are not perfectly indexed with the rate of inflation, which makes it $\frac{w}{r}$ be time function, and therefore cannot be constant, i.e., $b \neq 0$.

Production functions induced by the form of substitution elasticity. In this case, if the statistical data research confirms the linear dependence of the variable (σ), i.e.

$$\sigma = a + bk, \quad (25)$$

the differential equation is obtained:

$$\frac{R(k)}{kR'(k)} = a + bk, \quad (26)$$

whose solution is (particular case $b = 0$, and therefore $a = \sigma = const$):

$R(k) = Ak^{\frac{1}{\sigma}}$ (Here $R = MRS$). (27)

The following differential equation was thus obtained:

$$m\frac{f}{f'(k)} - k = Ak^{\frac{1}{\sigma}}, \quad (28)$$

the resolution of which leads to the known type function CES:

$$F(K, L) = A(\delta K^{-\varrho} + (1-\delta)L^{-\varrho})^{\frac{-m}{\varrho}}, \quad (29)$$

in which $\varrho = \frac{1-\sigma}{\sigma}$, respectively $\sigma = \frac{1}{1+\varrho}$.

It can be noted that in particular cases:

a) $\varrho = 0$, (respectively $\sigma = 1$) PF type CES is PF Cobb-Douglas;

b) $\varrho \to -1$, (respectively $\sigma \to \infty$) PF type CES is linear;

c) $\varrho \to \infty$, i.e. $\sigma = 0$ PF type CES is PF with fixed proportion of factors (Leontief PF).

The approach proposed in the communication may also be extended for more general cases where the dependencies of those characteristics are non-linear (parabolic, hyperbolic, logarithmic, exponential). It depends on the statistical data that identifies one or more of these non-linear functions.

Statistical data are discrete. The need to identify PF characteristics expressed in finite differences arises. It is proposed that the approximate value of the derivative of the $f(k)$, be calculated as follows: $f'(k) \approx \frac{f(k_{t+1})-f(k_t)}{k_{t+1}-k_t}$, and the other indicators to be calculated by applying the above formula.

4. Identification of the cost functions

Based on PF Cobb-Douglas and CES with total elasticity $\varepsilon = m > 1$ it can be noted, that the technologies of the examined economic systems have a expansion scale on the effect (increasing yield due to the learning and scale effect).

The economic content of PF can be obtained by identifying assumptions relating to the behavior of the economic system (firm). Two (more general) of these assumptions consists of maximizing profit and minimizing cost.

The standard system of assumptions, related to the purpose pursued by the company, is to minimize the cost:

- the company schedules a certain output level, which means that the endogenous variable Y for the PF becomes exogenous variable (argument) for the cost function;

- the market prices of output (supply) (p) and system input factors q_i are exogenous (information available to the company);

- the company chooses the input vector (the combination of) $X = (x_{1,} x_{2,} \ldots, x_n)$ which ensures the minimum cost level for achieving the programmed level set *Y*, and so this vector becomes endogenous. From these results the duality of the Cost Function in relation to the Production Function.

General Form of Cost Function (TC-total cost) is:

$$TC = FC + q_1 * x_1 + q_2 * x_2 + \cdots + q_n * x_n \,, \qquad (30)$$

$$FC - fixed\ cost\ (\ TC(0) = FC).$$

Variable cost $VC(y) = TC(Y) - FC$.

To identify the Cost function, the optimization issue is composed:

$$TC = FC + VC \ \rightarrow min,$$

subject to technological restriction*: F(X)=Y*, if all x_i are variables (in the case of the activity of the company for a long period). In the objective function of the formulated optimization problem, the term FC is constant, and therefore can be neglected.

The Khun-Tucker conditions leads us to the result:

$$x_j = A^{-\frac{1}{\varepsilon}} * \frac{\alpha_j}{q_j} * \left(\frac{q_i}{\alpha_i}\right)^{\frac{\alpha_i}{\varepsilon}} * y^{\frac{1}{\varepsilon}}, \qquad (31)$$

$$VC(q, Y) = A^{-\frac{1}{\varepsilon}} * \varepsilon * \prod_{i=1}^{n} \left(\frac{q_i}{\alpha_i}\right)^{\frac{\alpha_i}{\varepsilon}} * Y^{\frac{1}{\varepsilon}}. \qquad (32)$$

In case of PF type CES, applying the same scheme, it is:

$$x_j = A^{-\frac{1}{\varepsilon}} * Z^{\frac{1}{\rho}} * q_j^{-\frac{1}{1+\rho}} * \delta_j^{-\frac{1}{1+\rho}}, \qquad (33)$$

and the Variable Cost function:

$$VC(q, Y) = A^{-\frac{1}{\varepsilon}} * Z^{\frac{1}{1-\varepsilon}} * Y^{\frac{1}{\varepsilon}}, \qquad (34)$$

where

$$Z = \sum_{i=1}^{n} q_j^{1-\sigma} * \delta_i^{\sigma}, \sigma = \frac{1}{1+\rho}. \qquad (35)$$

5. Conclusions

The use of mathematical modeling in the production process aims to streamline economic activity. The correct specification, argued from a theoretical point of view, of the used functions is a very important step in the estimation process. The forms of the production function, presented in the article are induced by the marginal yields, the substitution elasticities, by the elasticities, the one that widens the possibilities of the researcher in the process of choosing it based on concrete statistical data. The small number of observations of macroeconomic indicators, which currently characterizes the Republic of Moldova, creates additional difficulties in estimating the production function used for practical needs, such as determining the Potential GDP (Toacă Z., Tolocico L. 2012). New forms of the production function widen the range of attempts to obtain a robust estimate of it. The specifications of the proposed production functions can be supplemented with an indicator, which will determine the influence of technical-scientific progress caused by the improvement of the workforce qualification and the modernization of technologies, dictated by the conditions of competition. The correctness of the specification of the production function is the basis of the conclusions, which determine the decisions taken at both microeconomic and macroeconomic level.

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