PORTFOLIO OPTIMIZATION - APPLICATION OF SHARPE MODEL USING LAGRANGE

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Abstract:
This paper presents the model developed by William Sharpe regarding the determination of the structure of the effective securities portfolio and the application of this model on the Romanian capital market. In this respect, the portfolio of shares used in our analysis is a portfolio of shares of the financial investment companies (SIF), listed on the Bucharest Stock Exchange (BVB), and for determining the structure of the efficient portfolio, there is built and minimized a function of type Lagrange. Also, to support practitioners, the paper also presents a series of mathematical demonstrations of variables used in modeling.

Keywords: modern portfolio theory, Sharpe model, lagrangian
JEL classification: C02, G11, G17

1. Introduction

This paper presents theoretically and applicably the model developed by William Sharpe regarding the determination of the structure of the efficient securities portfolio. The stock portfolio used in our analysis is a portfolio of shares of financial investment companies (SIF) listed on the Bucharest Stock Exchange (BVB), and for determining the structure of the efficient portfolio, there is built and minimized a function of type Lagrange. Also, to support practitioners, the paper also presents a series of mathematical demonstrations of variables used in modeling.

The fundamentals of the modern theory of the portfolio were put forward by Harry Markowitz and William Sharpe. In the sense of Markowitz, ”*The process of selecting a portfoliu may be divided into two stages. The firs*

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stage starts with beliefs about the future performances of available securities. The second stage starts with the relevant beliefs about future performances and with the choice of portfoliu." (Markowitz, 1952, pp.77). Sharpe, PhD student of Markowitz, is working out "*a simplified model of the relationships among securities, indicates the manner in which it allows the portfoliu analysis problem to be simplified, and provides evidence on the costs as well as the desirability of using the model for practical applications of the Markowitz technique.*"(Sharpe, 1963, pp.277).

The structure of the efficient portfolio, according to the two researchers mentioned above, is characterized by the highest profitability for a given level of risk or equivalent, the lowest risk for a given level of profitability. In other words, according to the return-risk criterion, investors consider themselves rational and pursue the lowest portfolio risk for a given level of expected return. "*Markowitz postulated that an investitor should maximize expected portfolio return* (μ_p) *while minimizing portfolio variance of return* (σ_p^2)" Rubinstein (2002, pp.1042).

The most important hypothesis of the portfolio's modern theory is that the profitability of each securities in the portfolio is in the form of a randomly distributed variable, characterized by average (expectation) and variance (risk). As a result, the expected return of the portfolio and portfolio risk are all of this kind.

2. Literature review

The first studies on the optimization of the securities portfolio were carried out by Markowitz (1952) and Roy (1952). Subsequently, the works of Tobin (1958); Treynor (1961); Sharpe (1963, 1964); Lintner (1965); Mossin (1966), made substantial contributions to the modern theory of the portfolio (see Holton, 2003, pp. 15-16). A successful work, in which the contributions of many authors to the modern theory of the portfolio are presented, is the work of Eltona and Gruber (1997), and a fifty-year retrospective of this theory is made by Rubinstein (2002).

The interest on this topic is quite high among researchers, and the applications of theory, in different markets, are many. Here are some of these works, as follows: Konno, Yamazaki (1991); Ledoit, Wolf (2003); Huang, Qiao(2012); Bilbao, Arenas, Jimenez, Gladish, Rodriguez(2006). We also mention here some articles written by Romanian authors on this issue, namely: Turcas, Dumiter, Brezeanu, Farcas, Coroiu (2017); Anghelache, Anghel

(2014); Badea, Petrescu, Stegaroiu, Ștefan (2010); Balteș, Dragoe (2015); Dima, Cristrea (2009); Anghelache G, Anghelache C (2014); Panait, Diaconescu (2012); Stancu, Predescu (2010); Zavera (2017).

It should be mentioned that in most of the works written by Romanian authors we can see the use of the Markowitz model or other models derived from the modern theory of the portfolio and almost no Sharpe model.

That being said, we are also making a very important statement, namely that one of the works that we consider reference in the importance and use of lagrangean in optimization problems is that of Fisher (2004).

3. Methodology

The Sharpe model provides, according to the investor's risk aversion degree, the structure of the efficient portfolio (PE), ie, the structure of the minimum risk portfolios for an expected return to be higher than the expected return of the minimum risk portfolio *(PR_{min})* which is located on the so-called efficiency border.

In this model, the return and the risk of the financial asset (expressed by variance) are given by the following formal expressions: $R_i = \alpha_i + \beta_i R_m + \varepsilon_i$, respectively: $\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{\varepsilon i}^2$, where: $\varepsilon_i \sim N(0, \sigma_{\varepsilon i}^2)$; $\beta_i = \frac{cov(R_i, R_m)}{\sigma^2(R_m)}$. Portfolio risk (expressed by variance) is given by the following formal expression: $\sum_{i=1}^{n} x_i^2 \sigma_{\varepsilon i}^2 + \beta_p^2 \sigma_m^2$.

To determine the structure of the efficient portfolio, the objective function (minimizing portfolio variance value and implicitly minimizing portfolio volatility) has three restrictions, respectively: $\sum_{i=1}^{n} x_i \beta_i = \beta_p$; $\sum_{i=1}^{n} x_i \mu_i = \mu_p$; $\sum_{i=1}^{n} x_i = 1$.

As a result, Lagrange function *(L)* has the following formal expression (Brătian; Bucur; Opreana, 2016, pp. 375-382):

$$L = \frac{1}{2}\sum_{i=1}^{n} x_i^2 \sigma_{\varepsilon i}^2 + \frac{1}{2}\beta_p^2 \sigma_m^2 + \lambda_1\left(\sum_{i=1}^{n} x_i \beta_i - \beta_p\right) + \lambda_2\left(\sum_{i=1}^{n} x_i \mu_i - \mu_p\right) + \lambda_3\left(\sum_{i=1}^{n} x_i - 1\right) \tag{1}$$

To minimize *L*, the optimal conditions are:

$$\begin{cases} \dfrac{\partial L}{\partial x_i} = 0 \\ \dfrac{\partial L}{\partial \beta_p} = 0 \\ \dfrac{\partial L}{\partial \lambda_1} = 0 \\ \dfrac{\partial L}{\partial \lambda_2} = 0 \\ \dfrac{\partial L}{\partial \lambda_3} = 0 \end{cases} \rightarrow \begin{cases} x_i \sigma_{\varepsilon i}^2 + \lambda_1 \beta_i + \lambda_2 \mu_i + \lambda_3 = 0 \\ \beta_p \sigma_m^2 + \lambda_1 (-1) = 0 \\ \sum_{i=1}^{n} x_i \beta_i + \beta_p (-1) = 0 \\ \sum_{i=1}^{n} x_i \mu_i = \mu_p \\ \sum_{i=1}^{n} x_i = 1 \end{cases} \tag{2}$$

The two above systems are equivalent because:

$$\frac{\partial L}{\partial x_i} = \frac{\partial}{\partial x_i}\left[{}^1\!/_2 \left(\sum_{i=1}^{n} x_i^2 \sigma_{\varepsilon i}^2 + \beta_p^2 \sigma_m^2\right) + \lambda_1 \left(\sum_{i=1}^{n} x_i \beta_i - \beta_p\right)\right] +$$

$$+ \frac{\partial}{\partial x_i}\left[\lambda_2 \left(\sum_{i=1}^{n} x_i \mu_i - \mu_p\right) + \lambda_3 \left(\sum_{i=1}^{n} x_i - 1\right)\right] =$$

$$= \frac{\partial}{\partial x_i}\left[{}^1\!/_2 \sum_{i=1}^{n} x_i^2 \sigma_{\varepsilon i}^2\right] + \frac{\partial}{\partial x_i}\left[\lambda_1 \left(\sum_{i=1}^{n} x_i \beta_i - \beta_p\right)\right] + \frac{\partial}{\partial x_i}\left[\lambda_2 \left(\sum_{i=1}^{n} x_i \mu_i -\right.\right.$$

$$\left.\left. - \mu_p\right)\right] + \frac{\partial}{\partial x_i}\left[\lambda_3 \left(\sum_{i=1}^{n} x_i - 1\right)\right] = x_i \sigma_{\varepsilon i}^2 + \left[\lambda_1 \left(\sum_{i=1}^{n} \frac{\partial}{\partial x_i} x_i \beta_i - 0\right)\right] +$$

$$+ \frac{\partial}{\partial x_i}\left[\lambda_2 \left(\sum_{i=1}^{n} x_i \mu_i - 0\right)\right] + \left[\lambda_3 \left(\sum_{i=1}^{n} \frac{\partial}{\partial x_i} x_i - 0\right)\right] = x_i \sigma_{\varepsilon i}^2 + \lambda_1 \beta_i + \lambda_2 \mu_i +$$

$$+ \lambda_3; \tag{3}$$

$$\frac{\partial L}{\partial \beta_p} = \frac{\partial}{\partial \beta_p}\left[{}^1\!/_2 \left(\sum_{i=1}^{n} x_i^2 \sigma_{\varepsilon i}^2 + \beta_p^2 \sigma_m^2\right) + \lambda_1 \left(\sum_{i=1}^{n} x_i \beta_i - \beta_p\right)\right] +$$

$$+ \frac{\partial}{\partial \beta_p}\left[\lambda_2 \left(\sum_{i=1}^{n} x_i \mu_i - \mu_p\right) + \lambda_3 \left(\sum_{i=1}^{n} x_i - 1\right)\right] = 0 + \beta_p \sigma_m^2 + \lambda_1 (0 - 1) +$$

$$+ 0 = \beta_p \sigma_m^2 + \lambda_1 (-1); \tag{4}$$

$$\frac{\partial L}{\partial \lambda_1} = \frac{\partial}{\partial \lambda_1}\left[{}^1\!/_2 \left(\sum_{i=1}^{n} x_i^2 \sigma_{\varepsilon i}^2 + \beta_p^2 \sigma_m^2\right) + \lambda_1\left(\sum_{i=1}^{n} x_i \beta_i - \beta_p\right)\right] +$$

$$+\frac{\partial}{\partial \lambda_1}\left[\lambda_2\left(\sum_{i=1}^{n} x_i \mu_i - \mu_p\right) + \lambda_3\left(\sum_{i=1}^{n} x_i - 1\right)\right] =$$

$$= \frac{\partial}{\partial \lambda_1}\left[{}^1\!/_2 \sum_{i=1}^{n} x_i^2 \sigma_{\varepsilon i}^2\right] + \frac{\partial}{\partial \lambda_1}\left[\lambda_1\left(\sum_{i=1}^{n} x_i \beta_i - \beta_p\right)\right] + \frac{\partial}{\partial \lambda_1}\left[\lambda_2\left(\sum_{i=1}^{n} x_i \mu_i -\right.\right.$$

$$\left.\left.-\mu_p\right)\right] + \frac{\partial}{\partial \lambda_1}\left[\lambda_3\left(\sum_{i=1}^{n} x_i - 1\right)\right] = 0 + \left(\sum_{i=1}^{n} x_i \beta_i - \beta_p\right) + 0 + 0 = \sum_{i=1}^{n} x_i \beta_i +$$

$$+\beta_p(-1)\,; \tag{5}$$

$$\frac{\partial L}{\partial \lambda_2} = \frac{\partial}{\partial \lambda_2}\left[{}^1\!/_2 \left(\sum_{i=1}^{n} x_i^2 \sigma_{\varepsilon i}^2 + \beta_p^2 \sigma_m^2\right) + \lambda_1\left(\sum_{i=1}^{n} x_i \beta_i - \beta_p\right)\right] +$$

$$+\frac{\partial}{\partial \lambda_2}\left[\lambda_2\left(\sum_{i=1}^{n} x_i \mu_i - \mu_p\right) + \lambda_3\left(\sum_{i=1}^{n} x_i - 1\right)\right] =$$

$$= \frac{\partial}{\partial \lambda_2}\left[{}^1\!/_2 \sum_{i=1}^{n} x_i^2 \sigma_{\varepsilon i}^2\right] + \frac{\partial}{\partial \lambda_2}\left[\lambda_1\left(\sum_{i=1}^{n} x_i \beta_i - \beta_p\right)\right] + \frac{\partial}{\partial \lambda_2}\left[\lambda_2\left(\sum_{i=1}^{n} x_i \mu_i -\right.\right.$$

$$\left.\left.-\mu_p\right)\right] + \frac{\partial}{\partial \lambda_2}\left[\lambda_3\left(\sum_{i=1}^{n} x_i - 1\right)\right] = 0 + 0 + \left(\sum_{i=1}^{n} x_i \mu_i - \mu_p\right) = \sum_{i=1}^{n} x_i \mu_i -$$

$$-\mu_p; \tag{6}$$

$$\frac{\partial L}{\partial \lambda_3} = \frac{\partial}{\partial \lambda_3}\left[{}^1\!/_2 \left(\sum_{i=1}^{n} x_i^2 \sigma_{\varepsilon i}^2 + \beta_p^2 \sigma_m^2\right) + \lambda_1\left(\sum_{i=1}^{n} x_i \beta_i - \beta_p\right)\right] +$$

$$+\frac{\partial}{\partial \lambda_3}\left[\lambda_2\left(\sum_{i=1}^{n} x_i \mu_i - \mu_p\right) + \lambda_3\left(\sum_{i=1}^{n} x_i - 1\right)\right] =$$

$$= \frac{\partial}{\partial \lambda_3}\left[{}^1\!/_2 \sum_{i=1}^{n} x_i^2 \sigma_{\varepsilon i}^2\right] + \frac{\partial}{\partial \lambda_3}\left[\lambda_1\left(\sum_{i=1}^{n} x_i \beta_i - \beta_p\right)\right] + \frac{\partial}{\partial \lambda_3}\left[\lambda_2\left(\sum_{i=1}^{n} x_i \mu_i -\right.\right.$$

$$\left.\left.-\mu_p\right)\right] + \frac{\partial}{\partial \lambda_3}\left[\lambda_3\left(\sum_{i=1}^{n} x_i - 1\right)\right] = \sum_{i=1}^{n} x_i - 1. \tag{7}$$

In matrix form, the last system above (2) is written as follows:

$$\begin{pmatrix} \sigma_{\varepsilon 1}^2 & 0 & \cdots & 0 & 0 & \beta_1 & \mu_1 & 1 \\ 0 & \sigma_{\varepsilon 2}^2 & \cdots & 0 & 0 & \beta_2 & \mu_2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_{\varepsilon n}^2 & 0 & \beta_n & \mu_n & 1 \\ 0 & 0 & \cdots & 0 & \sigma_m^2 & -1 & 0 & 0 \\ \beta_1 & \beta_2 & \cdots & \beta_n & -1 & 0 & 0 & 0 \\ \mu_1 & \mu_2 & \cdots & \mu_n & 0 & 0 & 0 & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ \beta_p \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \mu_p \\ 1 \end{pmatrix} \tag{8}$$

$$\Rightarrow$$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ \beta_p \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} \sigma_{\varepsilon 1}^2 & 0 & \cdots & 0 & 0 & \beta_1 & \mu_1 & 1 \\ 0 & \sigma_{\varepsilon 2}^2 & \cdots & 0 & 0 & \beta_2 & \mu_2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_{\varepsilon n}^2 & 0 & \beta_n & \mu_n & 1 \\ 0 & 0 & \cdots & 0 & \sigma_m^2 & -1 & 0 & 0 \\ \beta_1 & \beta_2 & \cdots & \beta_n & -1 & 0 & 0 & 0 \\ \mu_1 & \mu_2 & \cdots & \mu_n & 0 & 0 & 0 & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 & 0 & 0 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \mu_p \\ 1 \end{pmatrix} \tag{9}$$

The volatility (as a measure of risk) of efficient portfolio return (σ_{PE}) is determined as follows:

$$\sigma_{PE}=\sqrt{\left(x_1 x_2 \cdots x_n \beta_p\right)\cdot\begin{pmatrix}\sigma_{\varepsilon 1}^2 & 0 & \cdots & 0 & 0\\ 0 & \sigma_{\varepsilon 2}^2 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & \sigma_{\varepsilon n}^2 & 0\\ 0 & 0 & \cdots & 0 & \sigma_m^2\end{pmatrix}\cdot\begin{pmatrix}x_1\\ x_2\\ \vdots\\ x_n\\ \beta_p\end{pmatrix}} \tag{10}$$

Note: The efficiency border has as an extreme point the minimum risk portfolio (PR_{min}) and represents the total portfolios between PR_{min} and the one with $\frac{\partial\mu_p}{\partial\sigma_p^2}=0$. These portfolios between PR_{min} and $\frac{\partial\mu_p}{\partial\sigma_p^2}=0$ are called efficient portfolios (PE).

Lagrange function for the minimum risk portfolio *(PR_{min})* has the following formal expression (Dragotă, (coordinator), pp. 282-283):

$$L=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}x_i x_j \sigma_{ij}+\lambda_1\left(\sum_{i=1}^{n}x_i-1\right) \tag{11}$$

To minimize L, the optimal conditions are:

$$\begin{cases}\dfrac{\partial L}{\partial x_i}=0\\ \dfrac{\partial L}{\partial \lambda_1}=0\end{cases}\rightarrow\begin{cases}\sum_{j=1}^{n}x_j\sigma_{ij}+\lambda_1\cdot 1=0\\ \sum_{i=1}^{n}x_i=1\end{cases} \tag{12}$$

In the matrix expression, the above system (12) is written as follows:

$$\begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} & 1 \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ \lambda_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \tag{13}$$

$$\Rightarrow$$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ \lambda_1 \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} & 1 \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \tag{14}$$

The expected return *(μ_{PRmin})* is given by the following formal expression:

$$\mu_{PR_{min}} = (x_1 x_2 \ldots x_n) \cdot \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{pmatrix} \tag{15}$$

Volatility of return for the minimum risk portfolio (σ_{PE}) is determined as follows:

$$\sigma_{PR_{\min}} = \sqrt{(x_1 x_2 \ldots x_n) \cdot \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1n} & \sigma_{2n} & \cdots & \sigma_n^2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}} \tag{16}$$

4. Application of the Sharpe portfolio optimization model to the Romanian capital market

Next we apply the Sharpe portfolio optimization model using the methodology described above. In this respect, the data used in our analysis are the daily closing prices of the shares of the financial investment companies (SIF), listed on the Bucharest Stock Exchange (www.bvb.ro). The period of analysis is from 28.10.2016 to 31.10.2017, one year. Also, the BET-FI index is also analyzed in the above-mentioned period (www.bvb.ro).

Whereas the structure of the efficient portfolio is linked to a given expected return, greater than the expected return of the portfolio with the minimum risk, we will first approach the portfolio with minimum risk. After determining the profitability of the minimum risk portfolio, we will offer a higher expected return than this and determine the structure of the efficient portfolio using the Sharpe model.

That being said, with regard to the minimum risk portfolio *(PR_{min})* the following values were obtained:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ \lambda_1 \end{pmatrix} =$$

$$\begin{pmatrix} 7.80989\mathrm{E}-05 & 3.80434\mathrm{E}-05 & 1.69233\mathrm{E}-05 & 3.17162\mathrm{E}-05 & 3.06831\mathrm{E}-05 & 1 \\ 3.80434\mathrm{E}-05 & 0.000108351 & 2.15378\mathrm{E}-05 & 4.32253\mathrm{E}-05 & 3.25785\mathrm{E}-05 & 1 \\ 1.69233\mathrm{E}-05 & 2.15378\mathrm{E}-05 & 6.16578\mathrm{E}-05 & 2.54116\mathrm{E}-05 & 1.09261\mathrm{E}-05 & 1 \\ 3.17162\mathrm{E}-05 & 4.32253\mathrm{E}-05 & 2.54116\mathrm{E}-05 & 0.000106109 & 3.15441\mathrm{E}-05 & 1 \\ 3.06831\mathrm{E}-05 & 3.25785\mathrm{E}-05 & 1.09261\mathrm{E}-05 & 3.15441\mathrm{E}-05 & 9.4104\mathrm{E}-05 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ \lambda_1 \end{pmatrix} =$$

$$\begin{pmatrix} 15983.00326 & -4270.828601 & -4480.348956 & -2614.869071 & -4616.956636 & 0.21460009 \\ -4270.828601 & 12555.790220 & -2708.213013 & -3200.839161 & -2375.909444 & 0,76133428 \\ -4480.348956 & -2708.213013 & 13627.13954 & -4132.942575 & -2305.634991 & 0.424148573 \\ -2614.869071 & -3200.839161 & -4132.942575 & 12536.2312 & -2587.580395 & 0.077725342 \\ -4616.956636 & -2375.909444 & -2305.634991 & -2587.580395 & 11886.08147 & 0.207352649 \\ 0.21460009 & 0,76133428 & 0.424148573 & 0.077725342 & 0.207352649 & -3.5665E-05 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ \lambda_1 \end{pmatrix} = \begin{pmatrix} 21.46\% \\ 7.61\% \\ 42.41\% \\ 7.78\% \\ 20.74 \\ -3{,}56649E-05 \end{pmatrix}$$

As a result, the structure of the portfolio with minimum risk is the following: *SIF1 shares = 21,46%; SIF2 shares = 7,61%; SIF3 shares = 42,41%; SIF4 shares = 7,78%; SIF5 shares = 20,74%.*
For this structure, the expected return and the minimum risk are:

$$\mu_{PR_{min}} = (21.46\% \quad 7.61\% \quad 42.41\% \quad 7.78\% \quad 20.74\%) \cdot \begin{pmatrix} 0{,}14918\% \\ 0{,}17603\% \\ -0{,}01290\% \\ 0{,}13189\% \\ 0{,}11221\% \end{pmatrix} = 0{,}07346\%$$

Note: To the above expected return corresponds a yearly expected return of 18,51%.

$$\sigma^2_{PR_{min}} = (21.46\% \quad 7.61\% \quad 42.41\% \quad 7.78\% \quad 20.74\%) \cdot$$

$$\cdot \begin{pmatrix} 7.80989E-05 & 3.80434E-05 & 1.69233E-05 & 3.17162E-05 & 3.06831E-05 \\ 3.80434E-05 & 0.000108351 & 2.15378E-05 & 4.32253E-05 & 3.25785E-05 \\ 1.69233E-05 & 2.15378E-05 & 6.16578E-05 & 2.54116E-05 & 1.09261E-05 \\ 3.17162E-05 & 4.32253E-05 & 2.54116E-05 & 0.000106109 & 3.15441E-05 \\ 3.06831E-05 & 3.25785E-05 & 1.09261E-05 & 3.15441E-05 & 9.4104E-05 \end{pmatrix} \cdot \begin{pmatrix} 21{,}46\% \\ 7{,}61\% \\ 42{,}41\% \\ 7{,}78\% \\ 20{,}74\% \end{pmatrix} =$$

= 0,0000356649

$$\Rightarrow$$

$$\sigma_{PR_{min}} = \sqrt{\sigma^2_{PR_{min}}} = 0.597\%$$

Note: To the above volatility corresponds a yearly volatility of 9,48%.

Regarding the structure of the efficient portfolio *(PE)*, using μ_{PE}=0,09% (to this expected return corresponds an yearly expected return of 22,68%), the following values were obtained:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ \beta_p \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} =$$

$$= \begin{pmatrix} 0{,}00004234 & 0 & 0 & 0 & 0 & 0 & 1.08555034 & 0.001491778 & 1 \\ 0 & 0{,}00006302 & 0 & 0 & 0 & 0 & 1.222751782 & 0.00176029 & 1 \\ 0 & 0 & 0{,}00005183 & 0 & 0 & 0 & 0.57392146 & -0.00012905 & 1 \\ 0 & 0 & 0 & 0{,}00007234 & 0 & 0 & 1.057049704 & 0{,}001318855 & 1 \\ 0 & 0 & 0 & 0 & 0{,}00005803 & 0 & 1.091298705 & 0{,}001122071 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3.02436E-05 & -1 & 0 & 0 \\ 1.08555034 & 1.222751782 & 0.573592146 & 1.0570449704 & 1.091298705 & -1 & 0 & 0 & 0 \\ 0.001491778 & 0.00176029 & -0{,}00012905 & 0.001318855 & 0.001122071 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}^{-1} \cdot$$

$$\cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0{,}0900\% \\ 1 \end{pmatrix}$$

$$\Rightarrow$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ \beta_p \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} =$$

$$= \begin{pmatrix} 15184.5020 & -6634.7209 & -415.7786 & -4366.3689 & -4599.1908 & -1025.1481 & -0.0310 & 243.3744 & 0.0276 \\ -6634.7209 & 10291.1838 & 2776.7362 & -3257.8463 & -3175.3527 & 64.9900 & 0.0020 & 261.5079 & -0.1113 \\ 415.7786 & 2776.7362 & 1358.3907 & -1150.3591 & -3400.5463 & -301.2286 & -0.0091 & -581.5108 & 0.8598 \\ -4366.3689 & -3257.8463 & -1150.3591 & 11455.5192 & -2680.9450 & -199.9461 & -0.0060 & 79.1891 & 0.0733 \\ -4599.1908 & -3175.3527 & -3400.5463 & -2680.9450 & 13856.0349 & 1461.3328 & 0.0442 & -2.5607 & 0.1506 \\ -1025.1481 & 64.9900 & -301.2286 & -199.9461 & 1461.3328 & 177.2321 & -0.9946 & 331.3168 & 0.6289 \\ -0.0310 & 0.0020 & -0.0091 & -0.0060 & 0.0442 & -0.9946 & 0.0000 & 0.0100 & 0.0000 \\ 243.3744 & 261.5079 & -581.5108 & 79.1891 & -2.5607 & 331.3168 & 0.0100 & -28.1200 & 0.0208 \\ 0.0276 & -0.1113 & 0.8598 & 0.0733 & 0.1506 & 0.6289 & 0.0000 & 0.0208 & -0.0001 \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0{,}0900\% \\ 1 \end{pmatrix}$$

$$\Rightarrow$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ \beta_p \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 24.7\% \\ 12.4\% \\ 33.6\% \\ 14.5\% \\ 14.8\% \\ 0{,}9270733 \\ 0{,}00003 \\ -0{,}00454 \\ -0{,}00003 \end{pmatrix}$$

As a result, the structure of the portfolio with minimum risk is the following: *SIF1 shares* =24,7%; SIF2 *shares* = 12,4%; SIF3 *shares* = 33,6%; SIF4 *shares* = 14,5%; SIF5 *shares* = 14,8%.
For this structure, volatility is:

$$\sigma^2_{PR_{min}} = (24.7\% \quad 12.4\% \quad 33.6\% \quad 14.5\% \quad 14.8\% \quad 0{,}9270733) \cdot$$

$$\cdot \begin{pmatrix} 0{,}00004234 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0{,}00006302 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0{,}00005183 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0{,}00007234 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0{,}00005803 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3.02436\mathrm{E}-05 \end{pmatrix} \cdot \begin{pmatrix} 24{,}7\% \\ 12{,}4\% \\ 33{,}6\% \\ 14{,}5\% \\ 14{,}8\% \\ 0{,}9270733 \end{pmatrix} =$$

=0,000038194

$$\Rightarrow$$

$$\sigma_{PR_{min}} = \sqrt{\sigma^2_{PR_{min}}} = 0.618\%$$

Remarks:

1. To the above volatility corresponds an yearly volatility of: *9,81%.*

2. *coefficients β from the regression function are:* β_{SIF1} *=1,08555034;* β_{SIF2} *=1,22275178;* β_{SIF3} *= 0,57359215;* β_{SIF4} *=1,05704970;* β_{SIF5} *=1,09129871.*
3. *Variances of residues* $\left(\sigma_{\varepsilon i}^2\right)$ *are:* $\sigma_{\varepsilon sif1}^2$ *= 0,00004234;* $\sigma_{\varepsilon sif2}^2$ *=0,00006302;* $\sigma_{\varepsilon sif3}^2$ *= 0,00005183;* $\sigma_{\varepsilon sif4}^2$ *= 0,00007234;* $\sigma_{\varepsilon sif5}^2$ *=0,00005803;*
4. *The calculations were made in Excel.*

5. Conclusions

The results obtained by us, following the application of the Sharpe model on the Romanian capital market using Lagrange, can be synthesized as follows:

a) $\mu_{(yearly)PRmin} = 18{,}51\% \rightarrow \sigma_{(yearly)PRmin} = 9{,}48\% \rightarrow \begin{pmatrix} Sif1 = 21{,}46\% \\ Sif2 = 7{,}61\% \\ Sif3 = 42{,}41\% \\ Sif4 = 7{,}78\% \\ Sif5 = 20{,}74\% \end{pmatrix}$

b) $\mu_{(yearly)PE} = 22{,}68\% \rightarrow \sigma_{(yearly)PE} = 9{,}81\% \rightarrow \begin{pmatrix} Sif1 = 24{,}7\% \\ Sif2 = 12{,}4\% \\ Sif3 = 33{,}6\% \\ Sif4 = 14{,}5\% \\ Sif5 = 14{,}8\% \end{pmatrix}$

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